Minimum Impulse Coplanar Circle-Ellipse Transfer

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This report explores a technique devised by Contensou and Breakwell for the generation of optimal N-impulse coplanar orbit transfers. In this procedure, the problem is transformed so that the calculus of variations can be employed. Applying the technique to circle-ellipse transfer it is found that the properties of the solution family are derived in a straightforward, rigorous, and global manner, in sharp contrast with the earlier derivations. The extremal solutions fall into two classes: those that correspond to Hohmann-type transfers and those that correspond to three-point transfers with the middle point at an infinite distance from the center of attraction. For some final orbits, both classes provide candidates for the absolute minimum, and the better transfer can come from either class. For the other final orbits, only extremals that correspond to the three-point transfer exist.

Nomenclature

F = generalized Hamiltonian
F_1 = function F with ψ replaced by its optimal value
h = angular momentum
m = vehicular mass
r = radial coordinate of the vehicle
T = thrust
u = V/μ
V = characteristic velocity
x_i = state variables
β = argument of pericenter measured from a fixed reference line drawn through the focus
ϵ = orbital eccentricity
θ = polar angle of the vehicle
θ^* = true anomaly
λ_i = Lagrange multipliers
μ = gravitational parameter
ψ = angle between the thrust vector and the local horizontal
$(\ )_0$ = initial value of a variable
$(\dot{\ })$ = the time derivative

Introduction

BARRAR[1] has recently studied two-impulse transfer between ellipses of given geometry. He concluded that the least fuel is used if the ellipses are oriented so that they have the same argument of pericenter and if a Hohmann-type transfer is then performed. The transfer orbit then passes through apses of the terminal orbits and its apocenter coincides with the larger of the apocenters of the terminal orbits. Ting[2] has derived inequalities (presented in the Conclusions) that must hold if it is possible to reduce the fuel by use of a third impulse. He also demonstrated that any transfer employing more than three impulses can be replaced by a three-impulse transfer that consumes the same quantity of fuel.

Also, Breakwell[3] and Contensou[4] have shown recently that the problem of N-impulse transfer can be solved by the calculus of variations with orbital elements chosen as the state variables. This straightforward and powerful approach avoids the long, complex reasoning that the other investigators have used even when solving less general problems. In this paper, the calculus of variations technique is applied to the special case of circle-to-ellipse transfer in order to provide a check on the results of Barrar and Ting.

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Discussion

Euler-Lagrange Equations

We define a state vector x as $x_1 = h$, $x_2 = \epsilon$, and $x_3 = \beta$. The expressions for the time rates of change of these variables due to a thrust force have the form[3,5]

$$\dot{x}_i = (1/\mu)(T/m)f_i(x, \theta, \psi) \qquad i = 1, 2, 3$$

where

$$f_1 = h^2 \cos\psi/(1 + \epsilon \cos\theta^*)$$

$$f_2 = h\left[\sin\theta^* \sin\psi + \frac{2\cos\theta^* + \epsilon(1 + \cos^2\theta^*)}{1 + \epsilon\cos\theta^*}\cos\psi\right]$$

$$f_3 = \frac{h}{\epsilon}\left[-\cos\theta^* \sin\psi + \frac{2 + \epsilon\cos\theta^*}{1 + \epsilon\cos\theta^*}\sin\theta^* \cos\psi\right]$$

$$\theta^* = \theta - \beta$$

The time rate of change of the characteristic velocity is $\dot{V} = T/m$. Following Breakwell's analysis, we replace the independent variable time with u, defined by $u = V/\mu$. The system equations now have the form

$$dx_i/du = f_i(x, \theta, \psi) \qquad i = 1, 2, 3$$

The Lagrange multipliers of the calculus of variations obey the equations

$$d\lambda_i/du = -\partial F/\partial x_i \qquad i = 1, 2, 3$$

where

$$F(x, \lambda, \theta, \psi) = \sum_{i=1}^{3} \lambda_i f_i$$

The control variables are θ and ψ, and are chosen to maximize F. This implies

$$\partial F/\partial\theta = 0 \quad \text{and} \quad \partial F/\partial\psi = 0$$

Since F can be expressed as $F = A \sin\psi + B \cos\psi$, where A and B are independent of ψ, the equations governing the optimal ψ are

$$\sin\psi = A/(A^2 + B^2)^{1/2} \quad \text{and} \quad \cos\psi = B/(A^2 + B^2)^{1/2}$$

ψ may now be eliminated in the equation for F to obtain

$$F_1(x, \lambda, \theta) = (A^2 + B^2)^{1/2}$$

The optimal value of θ can be found only through a numerical search.

Because the system equations contain θ and β only through the function θ^*, we have

$$\frac{\partial F}{\partial \theta} = \frac{\partial F}{\partial \theta^*}\frac{d\theta^*}{d\theta} = \frac{\partial F}{\partial \theta^*} = 0$$

$$\frac{d\lambda_3}{du} = -\frac{\partial F}{\partial \beta} = -\frac{\partial F}{\partial \theta^*}\frac{d\theta^*}{d\beta} = \frac{\partial F}{\partial \theta^*} = 0$$

Thus λ_3 is constant.

Characteristics of the Extremal Family

The extremal family originating on the circular orbit with unitary angular momentum ($h_0 = 1$, $\epsilon_0 = 0$) was calculated using an IBM 7094 computer. λ_3 was fixed at zero. Thus the extremals were optimal with respect to variations in β and depended upon the single parameter $\lambda_{1_0}/\lambda_{2_0}$.

Numerical integration of the system and multiplier equations was employed rather than the analytic method suggested in Ref. 3. The integration proved to be quite stable for this particular problem.

The members of the extremal family are found to possess certain striking properties. Each extremal consists of subarcs along which θ, ψ, and $r = h^2/[\mu(1 + \epsilon \cos\theta^*)]$ are constant. Furthermore, the only values that θ^* and ψ ever assume are zero and π. This is possible because $\partial F_1/\partial\theta^*$ contains $\sin\theta^*$ as a factor.

Evidently each subarc corresponds to an impulse imparted to the vehicle being transferred. At the subarc junction the vehicle coasts in its current orbit to its next value of θ.

Since $\sin\theta^* = \sin\psi = 0$, $d\beta/du = 0$ whenever $\epsilon \neq 0$ and $1 + \epsilon \cos\theta^* \neq 0$. Instead of taking $\epsilon_0 = 0$ we can assume it to be an arbitrarily small positive value. Furthermore, each extremal is terminated when it returns to the

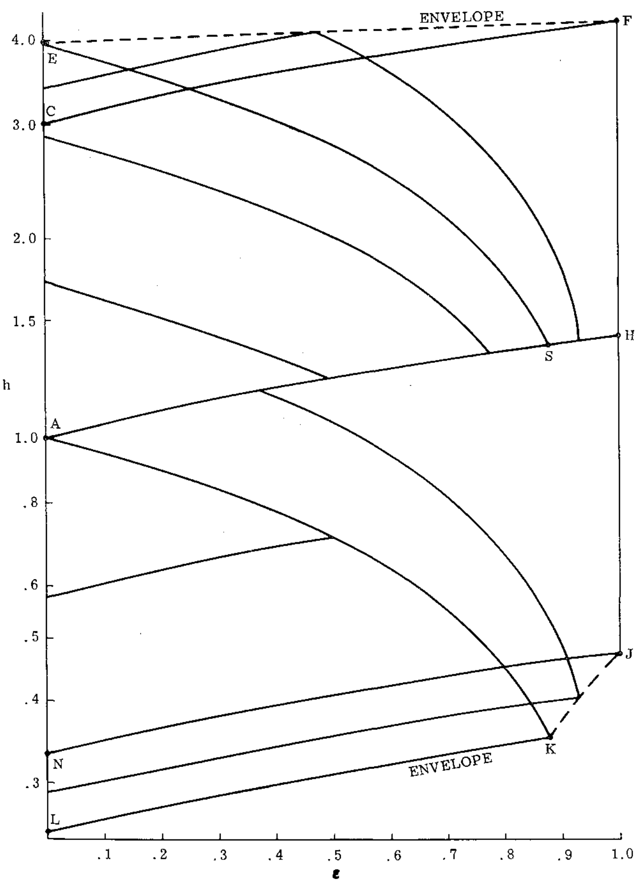

Fig. 1 Primary extremal family (semilog coordinates).

h axis. That β does not change when $1 + \epsilon \cos\theta^* = 0$ follows from

$$\frac{d\beta}{dh} = \frac{f_3}{f_1} = \frac{(2 + \epsilon \cos\theta^*) \sin\theta^*}{\epsilon h} = 0$$

Thus the extremals all lie in the $\beta = \beta_0$ plane. The value of β_0 is of course irrelevant.

Two-Point Transfers

The extremals at first follow one of two channels, AH or AK, of Fig. 1. Along AH, $\theta^* = \psi = 0$. If θ^* jumps to π before point S, the extremal will branch upward and reach the h axis between points A and E. If it is terminated before it reaches this axis, it corresponds to transfer from a circle to the apocenter of an entirely external ellipse (Fig. 2a). If it is terminated at the axis, it corresponds to a Hohmann transfer. We see that Hohmann transfers exist in our field only for final values of h up to point E.

If θ^* jumps to π between points S and H, the extremal branches upward until it reaches the envelope EF. EF is also a line of corners since there θ^* jumps back to zero and ψ jumps to π. The extremal continues until it reaches the h axis between points C and E. This third subarc is not minimal, however, because Jacobi's condition forbidding contact with an envelope has been violated.

If along the channel AH both θ^* and ψ jump to π, the extremal will branch downward. If it is terminated before it reaches the envelope JK it corresponds to transfer to the apocenter of an intersecting ellipse (Fig. 2b). At the envelope, θ^* jumps back to zero with ψ remaining at π. The extremal continues until it reaches the h axis between points L and N. As before, the third impulse is disqualified by the Jacobi condition.

Finally, we have the extremals that begin along the channel AK. Here, both θ^* and ψ are equal to π. The second subarc begins when θ^* jumps to zero. The extremal then moves to the left until it intersects the h axis between points A and L. The transfer represented is shown in Fig. 2c. Note that the subarc KL belongs to an extremal and also to the envelope. Thus the orbits corresponding to points on this portion of the envelope all have the same radius of pericenter.

Three-Point Transfers

Next, we consider the points above the envelope EF and below the envelope JKL. They are not covered by the extremal family, although transfers to these points can easily be constructed. What then is the optimal transfer?

The question is answered by constructing extremals analogous to the Goldschmidt solutions to the minimum-surface-of-revolution problem.[6] We note that when an extremal reaches point H (Fig. 1) and θ^* is chosen equal to π, then $dh/du = (dh/du)/(d\epsilon/du) = \pm\infty$. Thus, the extremal can move along the $\epsilon = 1$ ordinate without changing u. This entire ordinate is, therefore, a subarc of a transversal (line of constant u). In accordance with the transversality conditions of the calculus of variations, a horizontal Lagrange multiplier vector is associated with each point of the ordinate. This multiplier vector may have any magnitude but should point to the left. Extremals that are associated with these multiplier vectors and branch away from the points of the $\epsilon = 1$ channel will be designated "Goldschmidt" extremals. Note that the branch point is not determined by the choice of the initial multiplier vector at point A. In fact, the latter is identical for all "Goldschmidt" extremals since there is but one value of $\lambda_{1_0}/\lambda_{2_0}$ that can generate the subarc AH (Fig. 1).

Thus the following trajectory corresponds to the "Goldschmidt" solution. The vehicle is given an initial impulse (AH) that transfers it to a parabolic orbit. It then coasts to apocenter ($r = \infty$) and in this field-free region transfers with negligible expenditure of fuel to a different parabola.

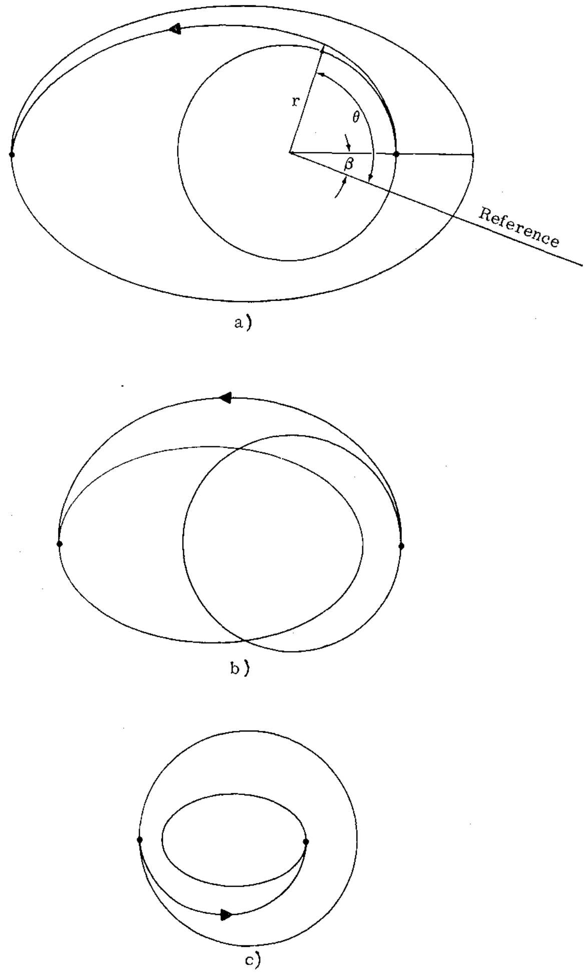

Fig. 2 Terminal and transfer orbits.

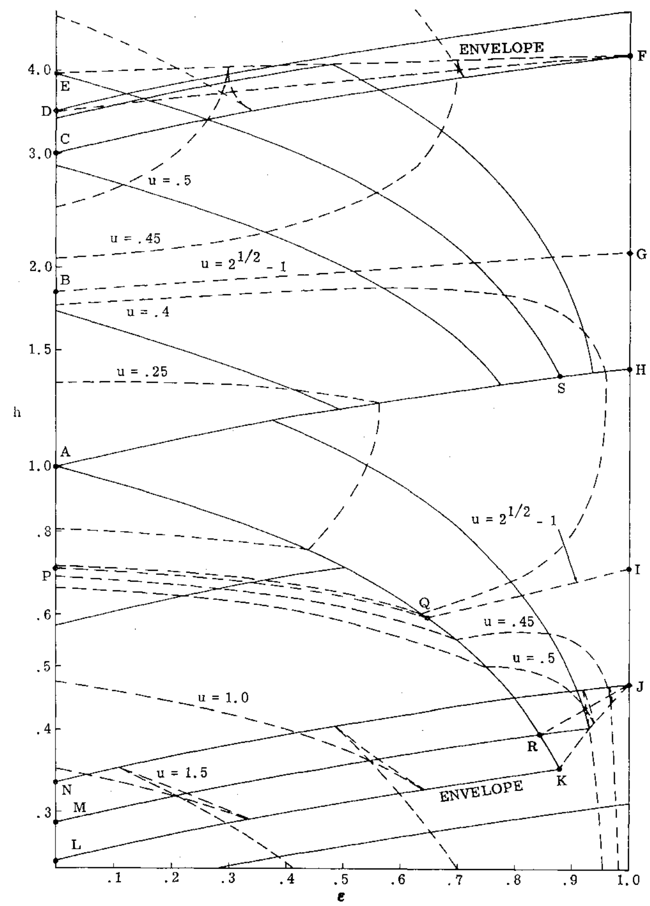

Fig. 3 Primary and "Goldschmidt" extremals with superimposed transversals (semilog coordinates).

The latter transfer is represented by a subarc along the $\epsilon = 1$ ordinate with $\theta^* = \pi$. At the pericenter of the new parabola another impulse is directed opposite to the vehicle's motion. It corresponds to a subarc that branches from the $\epsilon = 1$ channel and terminates on the h axis. The terminal orbit can be any of the types shown in Fig. 2.

Figure 3 displays the extremals of Fig. 1 plus those "Goldschmidt" extremals that branch away from the $\epsilon = 1$ channel above point F and below point J. Transversals have been superimposed on the extremals. We see that the extremals of the primary family and the "Goldschmidt" extremals require the same amount of fuel to reach endpoints on the arcs DF and MRJ (Fig. 3). In the regions between the arcs DF and EF and between the arcs JKL and JRM, the "Goldschmidt" extremals are more economical than those of the original family.

Conclusions

The Contensou-Breakwell variational equations for circle-ellipse transfers yield two classes of extremals. Those of the primary class possess the envelope $E\ F\ J\ K\ L$ of Fig. 1. In accordance with Jacobi's necessary condition, these extremals must be terminated at their points of contact with the envelope (i.e., their conjugate points). These extremals represent Hohmann-type transfers.

The extremals of the secondary class are analogous to Goldschmidt's solutions to the problem of the minimum surface of revolution. They represent three-point transfers with the middle point at $r = \infty$. These solutions exist for all terminal orbits. In the regions between arcs EF and DF, and between arcs MRJ and LKJ, extremals of the primary class provide a relative minimum, whereas those of the secondary class provide the absolute minimum.

The equations for the preceding subarcs are

EF $\quad (h^2 - \epsilon + 1)^3 - 2(3h^2 - \epsilon + 1)^2 = 0$

DF $\quad h - \dfrac{h^2 + \epsilon - 1}{(h^2 - \epsilon + 1)^{1/2}} + (\epsilon + 1)^{1/2} - 2^{1/2} = 0$

RJ $\quad (\epsilon + 1)^{1/2} - 2^{1/2}\epsilon - (h^2 - \epsilon + 1)^{1/2} + h = 0$

KJ $\quad (h^2 - \epsilon + 1)(\epsilon + 2)^2 - 2 = 0$

MR $\quad 11.94\,h^2 - \epsilon - 1 = 0$

LK $\quad 15.58\,h^2 - \epsilon - 1 = 0$

These equations can be derived with the aid of integrals furnished in Ref. 3; a derivation of equivalent equations is presented in Ref. 2.

When ϵ is set equal to zero in the equation for EF and the resulting cubic equation is solved for h^2, it is seen that at point E h^2 is approximately 15.58. A similar analysis of the equation for DF shows that, at point D, h^2 is approximately 11.94. For simplicity, these numbers have been introduced into the equations for MR and LK. These equations indicate that the Hohmann transfers cease to provide the absolute minimum when the radius ratio reaches 11.94, and that they are not even locally minimal when the ratio reaches 15.58. These conclusions verify those of Refs. 1, 2, and 7.

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Attitude Stability of Earth-Pointing Satellites

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One possible motion of an unsymmetrical rigid body in the gravitational field of a fixed particle is described as follows: the mass center of the body traces out a circular path centered at the particle; one of the body's centroidal principal axes of inertia remains normal to the plane of this orbit, and a second centroidal principal axis oscillates about the line joining the particle to the mass center of the rigid body. This investigation is concerned with the stability of such motions. It reveals that, not only the inertia properties of the body, but also the amplitude of the oscillations must be taken into account, and that the problem is essentially three-dimensional, i.e., that incorrect results are obtained when only planar motions are considered.

Introduction

EARTH-pointing satellites, i.e., satellites that always present the same, or nearly the same, face to the earth are of interest both because the moon is a satellite of this sort and because they are particularly well suited for use in space communications systems and as vehicles carrying earth observation equipment. The motion of the moon has, of course, been studied extensively. However, as this satellite differs markedly in certain respects from artificial satellites that can be constructed, such analyses as have been performed to describe the moon's behavior are not necessarily applicable to artificial bodies. For example, the stability conditions stated by Lagrange[1] and recently (independently) discussed by DeBra and Delp[2] were predicated on the assumption that attitude deviations from an "equilibrium configuration" are negligible, an assumption that may be tenable in the case of the moon, but that can be violated easily during injection into orbit of an artificial earth satellite. The stability question thus remains open.

The present paper is concerned with a problem in dynamics that has a direct bearing on this question. Specifically, a system comprised of a fixed particle P and an unsymmetrical rigid body B is considered, and stability criteria are developed for motions described as follows: the mass center P^* of B moves on a circular orbit while one of the three centroidal principal axes of B remains normal to the plane of the orbit, and a second principal axis oscillates with an amplitude θ_3^* about the line $P - P^*$. (This is a planar motion, but the analysis does *not* involve the assumption that the body is prevented from performing truly three-dimensional motions. On the contrary, the results obtained underscore the importance of taking the three-dimensional character of the problem into account.)

It is found that the stability of such motions depends both on the inertia properties of the body B and on the amplitude θ_3^*. If X_1, X_2, and X_3 are principal axes passing through the mass center of B, with X_3 normal to the orbit plane and X_1 oscillating about line $P - P^*$, and if I_1, I_2, and I_3 are the corresponding principal moments of inertia, two inertia parameters K_1 and K_2, defined as $K_1 = (I_2 - I_3)/I_1$ and $K_2 = (I_3 - I_1)/I_2$, may be used to characterize the body. Every motion of the kind under consideration can then be represented by a point in a three-dimensional space of the parameters K_1, K_2, and θ_3^*. A plane removed from such a space by choosing a particular value of θ_3^* becomes an "instability chart" when K_1 and K_2 are used as axes of a rectangular cartesian coordinate system and points representing unstable cases are identified. Figures 1 and 2 are examples of such charts; crosses designate unstable motions.

A comparison of Figs. 1 and 2 brings an important fact to light: for an amplitude as small as one degree, there

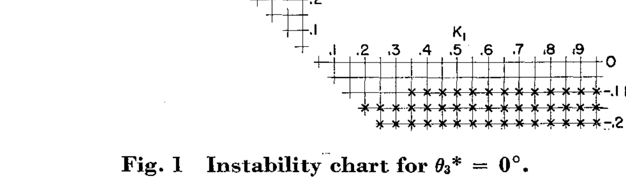

Fig. 1 Instability chart for $\theta_3^* = 0°$.

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